On the variational problem for the upper bounds of solute transport in double-diffusive convection

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Abstract

The formulation of the variational problems for the solute transport in a fluid layer in presence of double-diffusive thermal convection is discussed. It is shown that the variational functional obtained by Strauss can be generalized and the general functional leads to accurate upper bounds on the solute transport for the case of small and intermediate values of the Rayleigh number. The general functional however is a non-homogeneous one but for asymptotically large Rayleigh numbers it converges to the Strauss approximation. Thus for small and intermediate values of the Rayleigh numbers one should use the general functional and for vary large values of the Rayleigh numbers one can use the functional of Strauss.

Double-diffusive comvection is of considerable interest for oceanology as it is connected to the convective motion of water containing some amount of salt. In this case of convection in addition to the temperature gradient there is also a gradient of the salt concentration and this complicates the model equations. Below we discuss the variational problem for the upper bounds on the solute transport in a double diffusiove convection. Our discussion will be based on the optimum theory of turbulence which has been developed in the pioneering works of Howard, Busse, Doering and Constantin [1−4]. We shall not discuss the Doering-Constantin approach which easily leads to upper bounds if appropriate background fileds are chosen[5]. Instead of this our attention will be concentrated on a discussion of the work of Strauss[6] which is based on the Howard-Busse method which has many application to systems with thermal convection[7−11]. In order to avoid the complication of the analysis we shall not discuss the case of presence of rotation[12−16] and instead of this we shall follow the giudelines from [17,18].

Below we shall model the double-diffusive convection in a horizontal fluid layer by means of the equations of Boussinesq approximation[6,19]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho_0}\nabla p - \beta g S \vec{e}_z + \gamma g T \vec{e}_z + \nu \nabla^2 \vec{u} \tag{1}$$

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$$\frac{\partial \mathcal{S}}{\partial t} + \vec{u} \cdot \nabla \mathcal{S} = \kappa_S \nabla^2 \mathcal{S} \tag{2}$$

$$\frac{\partial \mathcal{T}}{\partial t} + \vec{u} \cdot \nabla \mathcal{T} = \kappa_T \nabla^2 \mathcal{T} \tag{3}$$

$$\nabla \cdot \vec{u} = 0 \tag{4}$$

where $\mathcal{S}$ is the concentration field which is divides as a horizontally averaged part $\overline{\mathcal{S}}$ [1] [2] and fluctuating part S

$$\mathcal{S}(x, y, z, t) = \overline{\mathcal{S}}(z) + S(x, y, z, t) \tag{5}$$

The same division is performed for the temperature field $\mathcal{T}$ too

$$\mathcal{T}(x, y, z, t) = \overline{\mathcal{T}}(z) + T(x, y, z, t) \tag{6}$$

In the above equations $\vec{u} = (u, v, w)$ is the fluid velocity (which is a fluctuating quantity as we assume that there is no mean flow: $\overline{\mathcal{U}} = 0$ [3]). ρ_0 is the average fluid density; p is the deviation of the pressure from hydrostatic pressure field which would exist if $\mathcal{S} = \overline{\mathcal{S}}$ and $\mathcal{T} = \overline{\mathcal{T}}$; γ is the coefficient of thermal expansion; β is the density change due to the unit change of

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[1] In this paper we shall use two kinds of averages:

1. Average over horizontal plane (horizontal average)

$$\overline{Q}(z, t) = \lim_{L \to \infty} \frac{1}{4L^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdy \;\; Q(x, y, z, t)$$

2. average over the volume of the layer (volume average)

$$\langle Q \rangle(t) = \lim_{L \to \infty} \frac{1}{4L^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} dxdydz \;\; Q(x, y, z, t)$$

[2] In additition we shall assume that we investigate the process of convection under quasistationary condition which is defined by the requirement that the horizontal averages of all quantities are time independent and the horizontal averages of all fluctuation parts of the quantities are 0. Because of this in the text below the horizontal and volume averages of the studied quantities depend only on z and the horizontal averages of the fluctuating parts of the studied quantities are set to 0.

[3] The basis of this assumption is in the nature of the convective motion in a fluid layer. When the heating of the layer is not very intensive the heat is transported through the layer by means of thermal conduction. Thermal conduction means that there is some temperature distribution in the fluid layer (i.e., $\overline{\mathcal{T}} \neq 0$ and the temperature fluctuations are zero) but there is no motion of fluid (i.e. the mean fluid velocity $\overline{\mathcal{U}} = 0$) and the fluctuations of the fluid velocity are 0 too. The arising of the convective motion of the fluid complicates the heat transfer. The heat is transferred not only by thermal conduction but also by thermal convection. This means that non-zero fluctuations arise and in addition the amplitude of these fluctuations can be large: they are nonlinear fluctuations which in general can not be treated by linear model equations. Because of this the obtained below model Euler-Lagrange integral-differential equations will be nonlinear.

solute concentration; g is the acceleration of the gravity; ν is the kinematic viscosity; κ_S is the solute diffusivity; κ_T is the thermal diffusivity and $\vec{e}_z$ is the unit vector in the direction opposite to the direction of gravity.

Below we shall obtain several integral-differential consequences of the Boussinesq equations. In the proces of obtaining of these relationships we shall use actively the condition of incompressibility of the fluid (the continuity equation (4)) and we the assumption that we study the system long after the last external influence of its dynamics, i.e., when the condition of quasistationarity is fulfilled that leads to independence of the horizontal averages of the flow quantitities on the times and in addition leads to the condition that the horizontal averages of any fluctuation quantities (and of combinations and derivatives of such quantities) are equal to 0.

Taking into account the above mentioned conditions after averaging (2) with respect to horizontal plane we obtain the relationship

$$\frac{d}{dz}\overline{w\mathcal{S}} = \kappa_S \frac{d^2\mathcal{S}}{dz^2} \tag{7}$$

The horizontal average of (3) leads to

$$\frac{d}{dz}\overline{w\mathcal{T}} = \kappa_T \frac{d^2\mathcal{T}}{dz^2} \tag{8}$$

The integration of (7) and (8) with respect to z leads to the relationships

$$\overline{wS} - \kappa_S \frac{d\mathcal{S}}{dz} = \text{const} \tag{9}$$

$$\overline{wT} - \kappa_T \frac{d\mathcal{T}}{dz} = \text{const} \tag{10}$$

The volume average of (9) and (10) leads to

$$\langle wS \rangle - \kappa_S \Delta S = \text{const} \cdot d \tag{11}$$

$$\langle wT \rangle - \kappa_T \Delta T = \text{const} \cdot d \tag{12}$$

where ΔS and ΔT are the differences in the salt concentration and temperature difference between the top and bottom borders of the fluid layer. As the integration constants are the same in (9) and (11) as well as in (10) and (12) we can eliminate the constants and in such a way we arrive at the two relationships

$$\overline{wS} - \kappa_S \frac{d\mathcal{S}}{dz} = \frac{\langle wS \rangle}{d} - \kappa_S \frac{\Delta S}{d} \tag{13}$$

$$\overline{wT} - \kappa_T \frac{d\mathcal{T}}{dz} = \frac{\langle wT \rangle}{d} - \kappa_T \frac{\Delta T}{d} \tag{14}$$

Below we shall perform non-dimensionalization of the quantities on the basis of the following units: ΔT will be the unit for temperature; ΔS will be the

unit for solute concentration; the layer thickness d will be the unit for length; d^2/κ_S will be the unit for time and κ_S/d will be the unit for velocity.

As next step we shall ontain two relationships known also as power integrals. Let us multiply (2) by S and average over the fluid layer. The non-dimensional result is

$$\langle wS \rangle^2 - \langle \overline{wS}^2 \rangle = \langle wS \rangle + \langle \mid \nabla S \mid^2 \rangle \tag{15}$$

The second power integral is obtained after multiplication of (1) by $\vec{u}$ averaging the obtained result over the fluid layer and non-dimansionalization of what is obtained. Thus we arrive at the relationship

$$- R\langle wS \rangle + \frac{\kappa_S}{\kappa_T} Ra \langle wT \rangle - \langle \mid \nabla \vec{u} \mid^2 \rangle = 0 \tag{16}$$

where $R = \frac{\beta g \Delta S d^3}{\nu \kappa_S}$ is the solute Rayleigh number and $Ra = \frac{\gamma g \Delta T d^3}{\nu \kappa_S}$ is the thermal Rayleigh number.

The goal of the methodology is the obtaining of upper bounds on the solute transport

$$Nu_S = 1 + \langle -wS \rangle \tag{17}$$

We shall write Nu_S in the form $Nu_S = 1 + \frac{1}{\mathcal{F}}$, i.e., when Nu_S has a maxumum $\mathcal{F}$ will have a minimum. In order to come to such a form of the relationship for the solute transport we rescale w, S, and $\theta' = \frac{\kappa_S}{\kappa_T} T$ as follows

$$w = A\hat{w}; \quad S = B\hat{S}; \quad \theta' = A\hat{\theta} \tag{18}$$

where A and B can be determined by substitution of (18) in the two power integrals above. The result of this is

$$B = A \frac{Ra\langle \hat{w}\hat{\theta} \rangle - \langle \mid \nabla \vec{\hat{u}} \mid^2 \rangle}{R \langle \hat{w}\hat{S} \rangle} = \alpha A \tag{19}$$

$$A = \left[\frac{\langle \hat{w}\hat{S} \rangle + \alpha \langle \mid \nabla \hat{S} \mid^2 \rangle}{\alpha (\langle \hat{w}\hat{S} \rangle^2 - \langle \overline{\hat{w}\hat{S}}^2 \rangle)} \right]^{1/2} \tag{20}$$

Now after some calculations we obtain the following relationship for $\mathcal{F}$ (we have set $S = \tilde{S}/R$ and in the final result below we omit the tilde and the hat signs)

$$\mathcal{F} = \frac{\lambda \langle \mid \nabla S \mid^2 \rangle [\langle \mid \nabla \vec{u} \mid^2 \rangle - Ra\langle w\theta \rangle] + (\langle \overline{wS} \rangle - \langle wS \rangle)^2}{\langle wS \rangle^2} \tag{21}$$

On the basis of (21) we shall formulate the variational problem below. But before this we shall obtain the particular case of (21) discussed by Strauss[6]. In this particular case there is a relationship between w and θ and because

of this the variational problem becomes simpler and the number of the corresponding Euler - Lagrange equations decreases. In order tho obtain the relationship between the two above-mentioned fields we start from Eq. (3) perform a non-dimensionalization and discuss the quasi-stationary approximation. What is obtained is

$$\tau w \frac{d\overline{\mathcal{T}}}{dz} + \tau \vec{u} \cdot \nabla T = \frac{d^2\overline{\mathcal{T}}}{dz^2} + \nabla^2 T \tag{22}$$

where $\tau = \kappa_S/\kappa_T$. Here we shall use an approximate relationship for $\frac{d\overline{\mathcal{T}}}{dz}$ which will lead to the approximation used by Strauss. We start from (14) perform a non-dimensionalization and obtain

$$\frac{d\overline{\mathcal{T}}}{dz} = 1 - \frac{\kappa_S}{\kappa_T}[\langle wT \rangle - \overline{\langle wT \rangle}] \tag{23}$$

Now the assumption of Strauss is that $\tau = \kappa_S/\kappa_T << 1$ and that the expression in [...] of Eq. (23) is not larger than 1 for any z. Then

$$\frac{d\overline{\mathcal{T}}}{dz} \approx 1 \tag{24}$$

and after the approximation $\tau \vec{u} \cdot \nabla T \approx 0$ the Eq. (22) can be reduced to

$$\tau w = \nabla^2 T \tag{25}$$

and because of the fact that $\theta' = T/\tau$ this lead to

$$w = \nabla^2 \theta' \tag{26}$$

Let $\theta = Ra\theta'$ and we consider the particular case of $1-\alpha$ solution of the variational problem

$$w = w(z)f(x,y); \theta = \theta(z)f(x,y); \qquad \nabla_1^2 f = -\alpha^2 f \tag{27}$$

Then from (26) we obtain

$$\theta(z) = -\frac{w(z)Ra}{\alpha^2} \tag{28}$$

Then the relationship for $\mathcal{F}$ in the approximation of Strauss is

$$\mathcal{F} = \frac{\lambda \langle \mid \nabla S \mid^2 \rangle [\langle \mid \nabla \vec{u} \mid^2 \rangle + \frac{Ra}{\alpha}\langle w^2 \rangle] + (\langle \overline{wS} \rangle - \langle wS \rangle)^2}{\langle wS \rangle^2} \tag{29}$$

Note that the functional of Strauss (29) is homogeneous one which will lead to homogeneous Euler-Lagrange equations and will simplify their numerical and analytical asymptotic solutions.

Let us now discuss a more general approximation and its consequences for the variational problem. let us again assume $\tau \vec{u} \cdot \nabla T \approx 0$ and let us make Eq.(8) dimensionless. The result is

$$\tau \frac{d}{dz} \overline{wT} = \frac{d^2 \mathcal{T}}{dz^2} \tag{30}$$

Thus after a substitution of Eqs. (23) and (30) in Eq. (22) we obtain the following relationship

$$\tau w \{1 - \tau [\langle wT \rangle - \overline{wT}]\} = \tau \frac{d}{dz} \overline{wT} + \nabla^2 T \tag{31}$$

Now we see that one can obtain the approximation of Strauss if one neglects the terms of order $O(\tau^2)$ in the left-hand side of Eq. (31) and in addition one has to make so-called internal layer approximation in the right-hand side of Eq.(31). The internal layer approximation means that one has to assume that $\frac{d}{dz}\overline{wT} \approx 0$ in the fluid layer. Strictly speaking the internal layer approximation is valid only in the internal sub-layer of the fluid layer. Up to some extent it is valid in the two intermediate layers of the fluid layer [4]. The internal layer approximation is not valid in the boundary layers of the fluid where all the fields (velocity, temperature and concentration) have to adjust their values to the boundary conditions on the borders of the fluid layer. When the Rayleigh numbers connected to the discussed problem have small and intermediate values then the thickness of the boundary layers of the fluid layer is large and because of this the approximation of Strauss is very crude. But when the values of the Rayleigh numbers become larger and larger then the boundary layers of the different fields become thinner and thinner and the approximation of Strauss can became reasonable one.

On the basis of all this we can formulate the variational problem for the upper bounds on the double-diffusive convection as follows

Case of arbitrary values of the Rayleigh numbers (Variational problem 1)

Given Ra and $\lambda > 0$, find the minimum $M(\lambda)$ of the functional:

$$\mathcal{F} = \frac{\lambda \langle \mid \nabla S \mid^2 \rangle [\langle \mid \nabla \vec{u} \mid^2 \rangle - Ra \langle w\theta \rangle] + (\langle \overline{wS} \rangle - \langle wS \rangle)^2}{\langle wS \rangle^2} +$$
$$\mu \left\{ w - \frac{1}{Ra} \nabla^2 \theta - \frac{\tau}{Ra} \overline{w\theta} + \frac{\tau^2}{Ra} w [\langle w\theta \rangle - \overline{w\theta}] \right\}$$

[4] The intermediate layers of the fluid layer are layers where the velocity field, the temperature field, and the concentration field make a transition from their almost constant values in the internal layer of fluid to the sharply depending on the coordinate values in the boundary layers of the fluid.

within the class of fields satisfying the boundary conditions and the continuity equation $\nabla \cdot \vec{u} = 0$.
Above μ is a Lagrange multiplier by means of which one takes into an account Eq.(31).

For asymptotic large values of the Rayleigh numbers we can use the approximation of Strauss:

Case of asymptotic large values of the Rayleigh numbers (Variational problem 2 - approximation of Strauss)

Given Ra and $\lambda > 0$*, find the minimum* $M(\lambda)$ *of the functional:*

$$\mathcal{F} = \frac{\lambda \langle \mid \nabla S \mid^2 \rangle [\langle \mid \nabla \vec{u} \mid^2 \rangle + \frac{Ra}{\alpha} \langle w^2 \rangle] + (\langle \overline{wS} \rangle - \langle wS \rangle)^2}{\langle wS \rangle^2}$$

within the class of fields satisfying the boundary conditions and the continuity equation $\nabla \cdot \vec{u} = 0$.

As concluding remarks we make several notes with respect to the two variational problems.

- The more general variational problem 1 is a non-homogeneous one. Because of this the corresponding Euler-Lagrange equations will be non-homogeneous too which means that the task for the obtaining upper bounds on the solute transport will be very difficult. However the obtained bounds will be very accurate especially for the case of small values of the Rayleigh numbers where the Strauss approximation is relatively inaccurate because of the large thickness of the boundary layers of the fields of the velocity, temperature and concentration.

- For large values of the Rayleigh numbers the upper bounds connected to the more general variational problem 1 will come close to the upper bounds of the less general variational problem 2 of Strauss and then the more simple and homogeneous Euler - Lagrange equation connected to the variational problem of Strauss can be used for calculation of the upper bounds.

- The variational problem of Strauss is derived only for the simplest possible form of the studied fields: fields with one characteristic wavenumber. The more general variational problem 1 is suitable for investigation of bounds for optimum fields that have arbitrary number of characteristic wave-numbers (so called multi-wave-number solutions of the Euler - Lagrange equations.

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